

1.3 Solving Systems of Linear Equations: Gauss-Jordan Elimination and Matrices

We can represent a system of linear equations using an **augmented matrix**. In general, a matrix is just a rectangular array of numbers. Working with matrices allows us to not have to keep writing the variables over and over.

To write a system of equations as an augmented matrix, line up all the variables on one side of the equal sign with the constants on the other side. Then, form a matrix of numbers with just the coefficients and constants from the equations.

Write the following systems using augmented matrices.

$$\begin{array}{rcl}
 3x - 4y = 12 & -3x + 2y - 6z = 14 & 2x - 4y = 9 \\
 y = 2x + 4 & 4x - 5y + 2z = -3 & -2y - z = 0 \\
 & 7x + 9y + 3z = 14 & x + y + 4 = 2z \\
 & & 3y + z = 5x
 \end{array}$$

If a matrix has m rows and n columns, we say the size of the matrix is $m \times n$. What are the sizes of the above matrices?

In order to solve a system, we want to “reduce” the augmented matrix to a form where we can easily identify the solution. This form is called “reduced-row echelon form.” It is equivalent to the original system, but simplified. The method by which we simplify an augmented matrix to its reduced form is called the Gauss-Jordan Elimination Method.

After reducing the 2nd augmented matrix above using the Gauss-Jordan Method, we would obtain the matrix shown. The solution can now be easily found by rewriting each row as an equation. What is the solution to the system?

$$\left[\begin{array}{ccc|c} -3 & 2 & -6 & 14 \\ 4 & -5 & 2 & -3 \\ 7 & 9 & 3 & 14 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array} \right]$$

There are 3 row operations that are used in the Gauss-Jordan Method:

1. Interchange any two rows. (This one is not used very often.)
Notation: $R_i \leftrightarrow R_j$ means to interchange row i with row j .
2. Replace any row by a nonzero constant multiple of itself.
Notation: $cR_i \rightarrow R_i$ means to replace row i with c times row i .
3. Replace any row by the sum of that row and a constant multiple of another row.
Notation: $R_i + aR_j \rightarrow R_i$ means to replace row i with the sum of row i and a times row j .

Example: Perform the given row operations in succession on the matrix.

$$\left[\begin{array}{ccc|c} 4 & 12 & 4 & 12 \\ 3 & 8 & 3 & 7 \\ -2 & 3 & -1 & 10 \end{array} \right] \frac{1}{4}R_1 \rightarrow R_1$$

$$R_2 - 3R_1 \rightarrow R_2$$

$$R_3 + 2R_1 \rightarrow R_3$$

Gauss-Jordan Elimination Method:

1. Write the system as an augmented matrix.
2. Look at the first entry in the first row. Make this entry into a 1 and all other entries in that column 0s. This is called **pivoting** the matrix about this element. (Note: If the entry is a 0, you must first interchange that row with a row below it that has a nonzero first entry.)
3. Once this is done, move down the diagonal to the second entry of the second row and **pivot** about this entry.
4. Continue until the whole matrix is in row-reduced form.

Solve the following system of equations using the Gauss-Jordan method:

$$-6x - 12y = 6$$

$$-4x - 5y = 10$$

Example: Perform the first pivot for the matrix below.

$$\left[\begin{array}{cc|c} \boxed{2} & 4 & 8 \\ 3 & 1 & 2 \end{array} \right]$$

Write down the row operations needed to perform the next pivot. $\left[\begin{array}{ccc|c} 1 & 3 & -1 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & -4 & -1 & 7 \end{array} \right]$.

The calculator command *rref* will reduce the matrix for you without having to do algebra. To enter a matrix into your calculator and find the row-reduced form:

- Press 2nd x^{-1} . (The TI-83 has a separate MATRX button.)
- Cursor over to EDIT and select where you want to store your matrix. Enter the size of the matrix and then enter in the matrix values.
- Go back to your home screen. Press MATRX and then cursor right to MATH. Scroll down and select *B:rref*.
- Select the matrix you want to reduce by pressing MATRX and then choosing your matrix under the NAMES column. Then close your parentheses and press enter. Note: The *rref* command only works when the number of rows is less than or equal to the number of columns.

Verify the solution we found before to the system below using *rref*.

$$\begin{aligned} -6x - 12y &= 6 \\ -4x - 5y &= 10 \end{aligned}$$

Systems of equations are used to model problems. Clearly define your variables so that you answer the question being asked in the problem. For the following word problems, set up a system of equations to solve the problem and then solve.

Example 1: (2.1, #24, Tan):

A theater charges \$3 for children and \$7 for adults. At a given show there were three times as many adults as children and the theater brought in revenue of \$5500, how many children and adults attended the show?

Example 2: Ben has a total of \$30,000 that he wants to invest in three accounts that have yields of 8%, 10%, and 11% interest per year, respectively. The amount of money invested in the 8% and 11% accounts combined is to equal the amount of money invested at 10%. If the investment goal is to have a return of 9.6% of the total investment, find how much Ben needs to invest in each account.

Example 3: A person has 91 coins made up of pennies, nickels, dimes, and quarters. The total value of the coins is \$6.25 and there are 4 times as many nickels as quarters. Additionally, the value in quarters is 12 times the value in pennies. How many of each type of coin does this person have? Let p , n , d , and q be the number of pennies, nickels, dimes, and quarters the person has respectively.

Example 4: A furniture company makes loungers, chairs, and footstools. The number of units of fabric and the amount of time (in minutes) needed to make each item are given in the table below. The company made half as many chairs as they did loungers and footstools combined. How many of each product were made if there were 652 units of fabric and 17 labor-hours available and all resources are used? Let x , y , and z represent the number of loungers, chairs, and footstools made respectively.

	Lounger	Chair	Footstool
Fabric	30	20	8
Time	45	30	15

Example to try on your own: A sporting goods store sells footballs, basketballs, and volleyballs. A football costs \$35, a basketball costs \$25, and a volleyball costs \$15. On a given day, the store sold 5 times as many footballs as volleyballs. They brought in a total of \$3750 that day, and the money made from basketballs alone was 4 times the money made from volleyballs alone. How many footballs, basketballs, and volleyballs were sold?

1.4 Systems of Equations with No Solution or Infinitely Many Solutions

Examples: Find the solutions represented by the following row-reduced matrices. Assume that the variables used in the systems of equations these matrices represent are x , y , and z .

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

When a system has infinitely many solutions, we **parameterize** the solution (write the solution in parametric form):

A particular solution is a specific solution to the system. What are some particular solutions for this system?

Solve the following system of equations:

$$x + 2y + 3z = 2$$

$$-3x + 3y = -1$$

$$7x - 13y + 7z = -1$$

$$-x + 7y + 4z = 3$$

Examples: The following reduced matrices represent systems of equations with variables x, y, z , and, if necessary, u . Determine the solutions to these systems.

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -5 \\ 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$